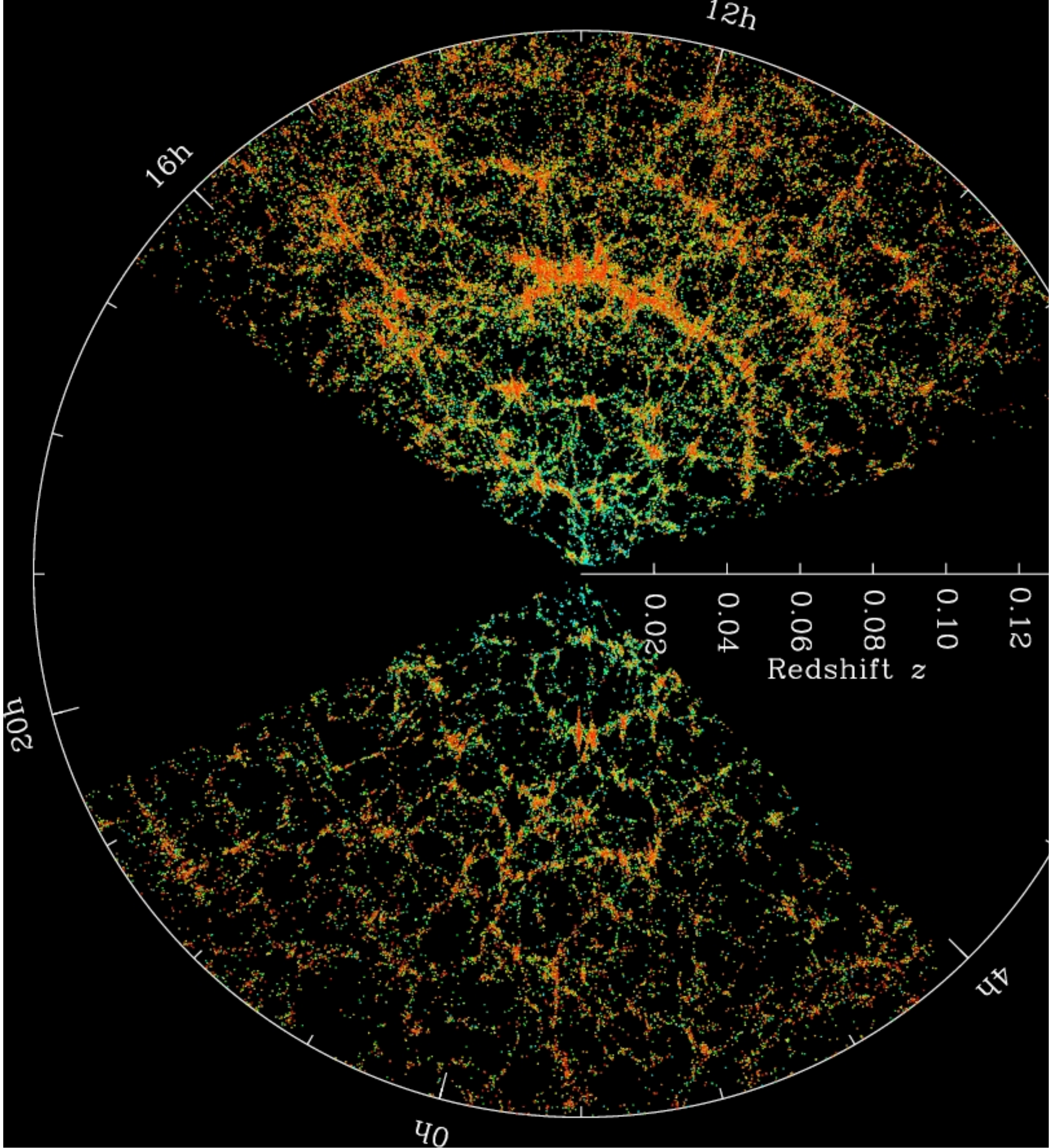
Francis Bernardeau

La structuration des galaxies, théorie



Introduction

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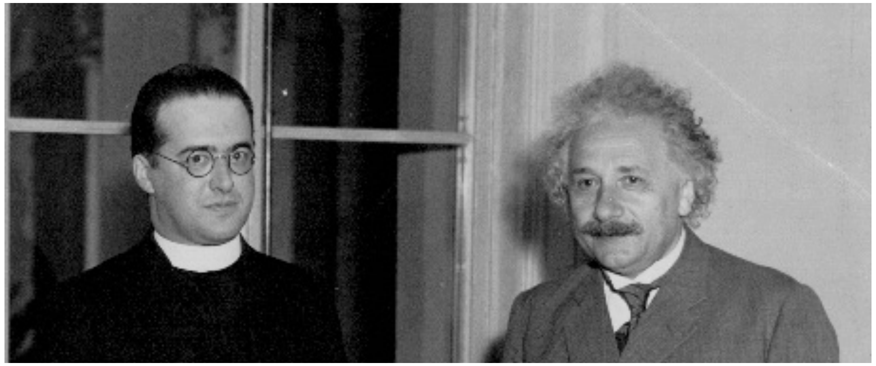

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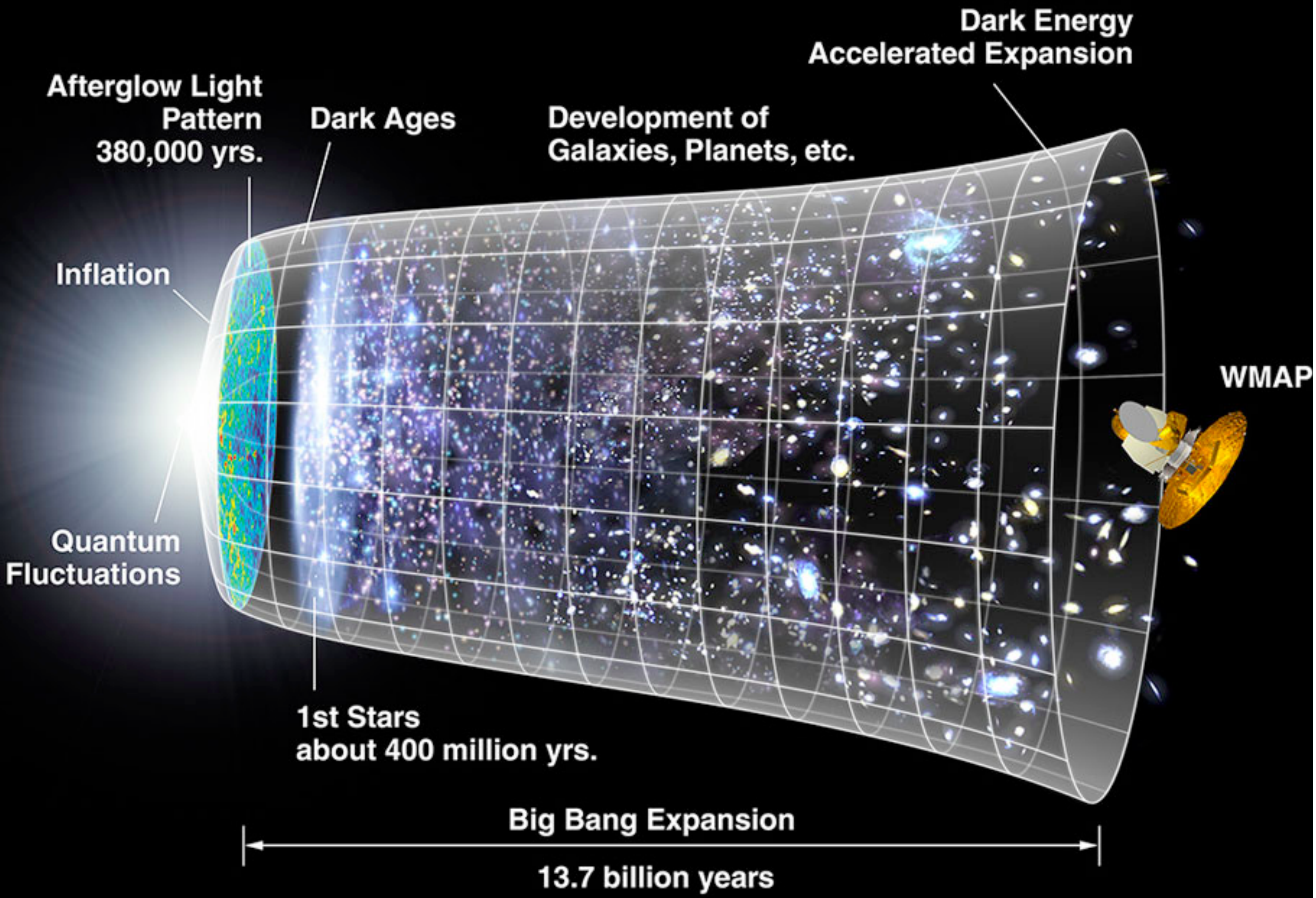
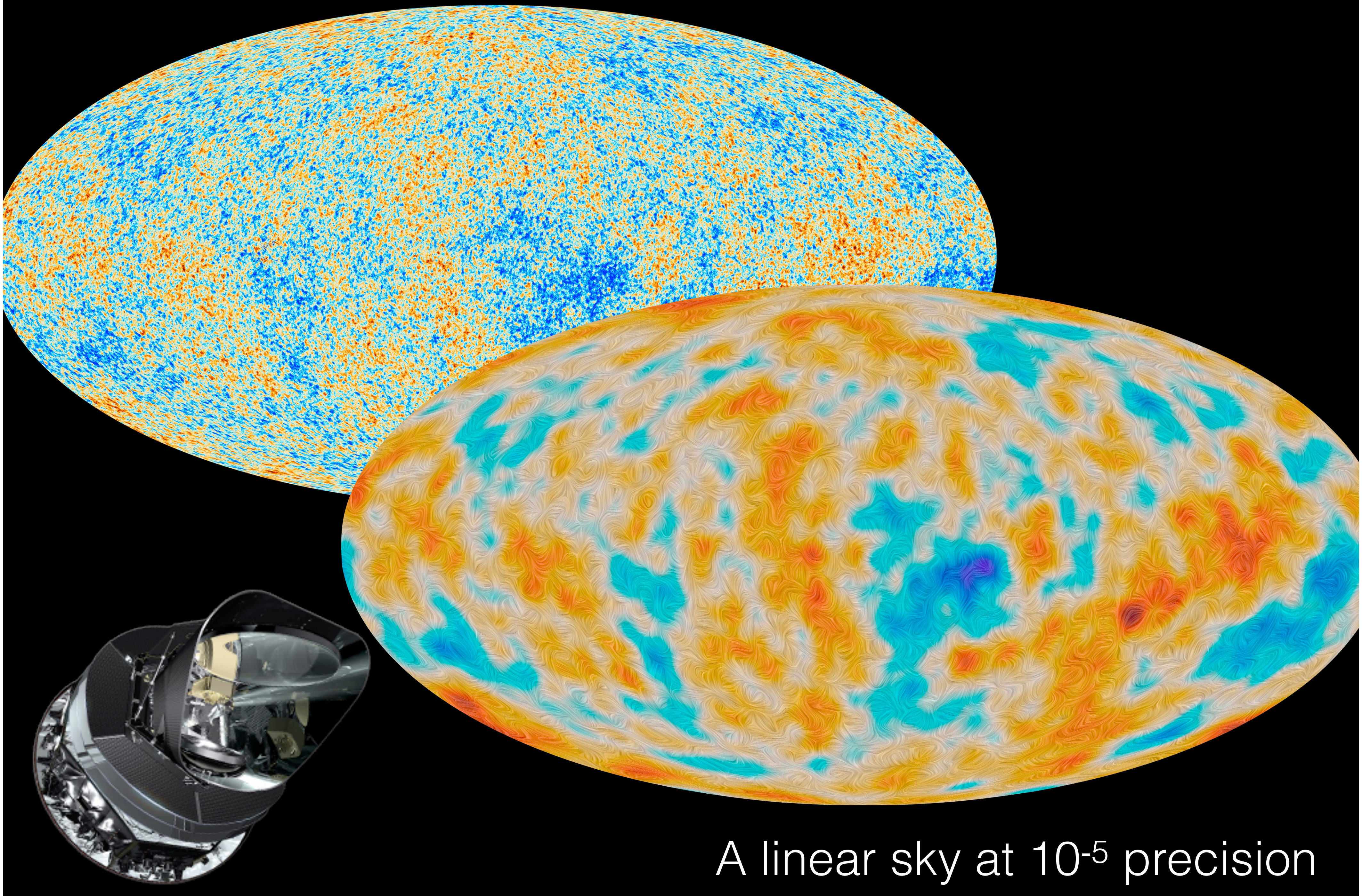
Observing the LSS of the universe



The CMB sky, temperature and polarization



A linear sky at 10^{-5} precision

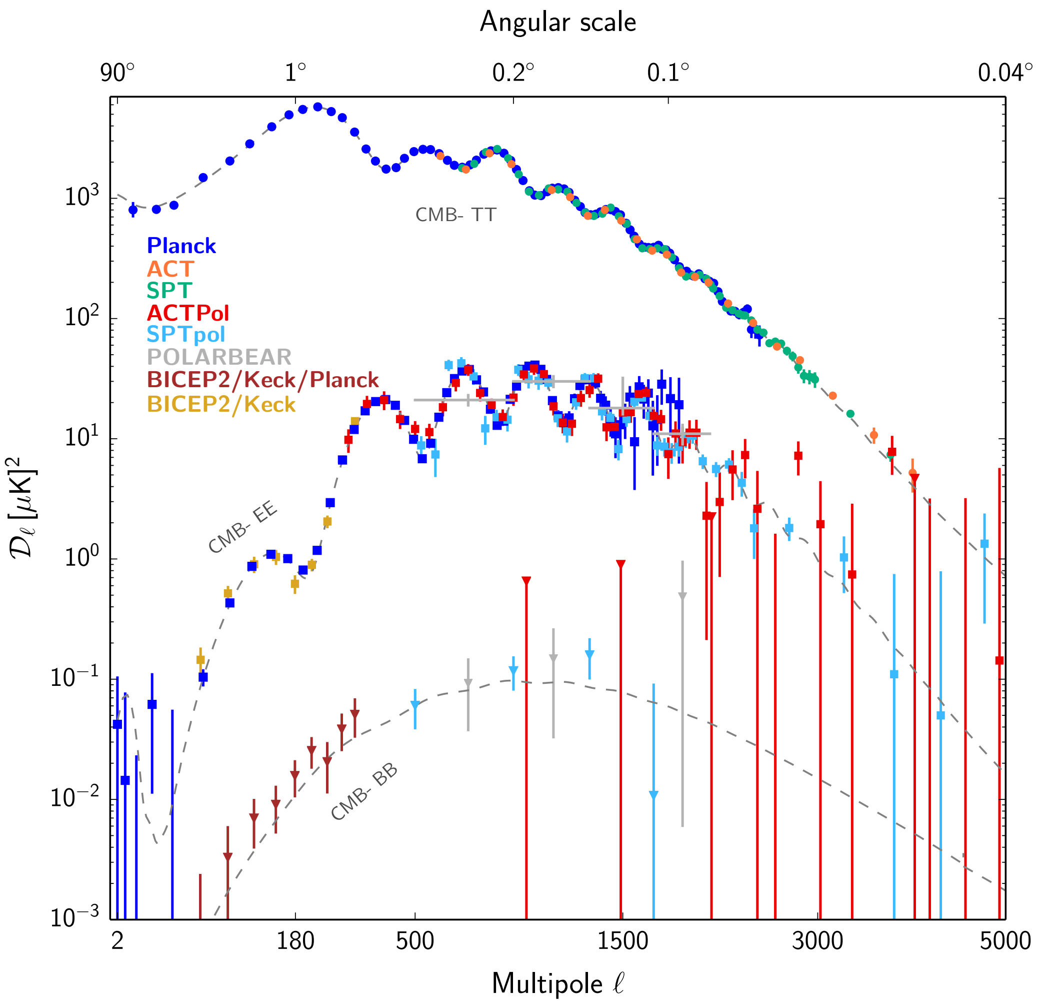
The linear sky

$$ {}_s a_{\ell m} = \int \mathrm{d}k \ {}_s\mathcal{T}_{\ell m}(k)\ \phi_{\rm adiab}(k) + \ldots $$

harmonic modes (for temperature and polarization)

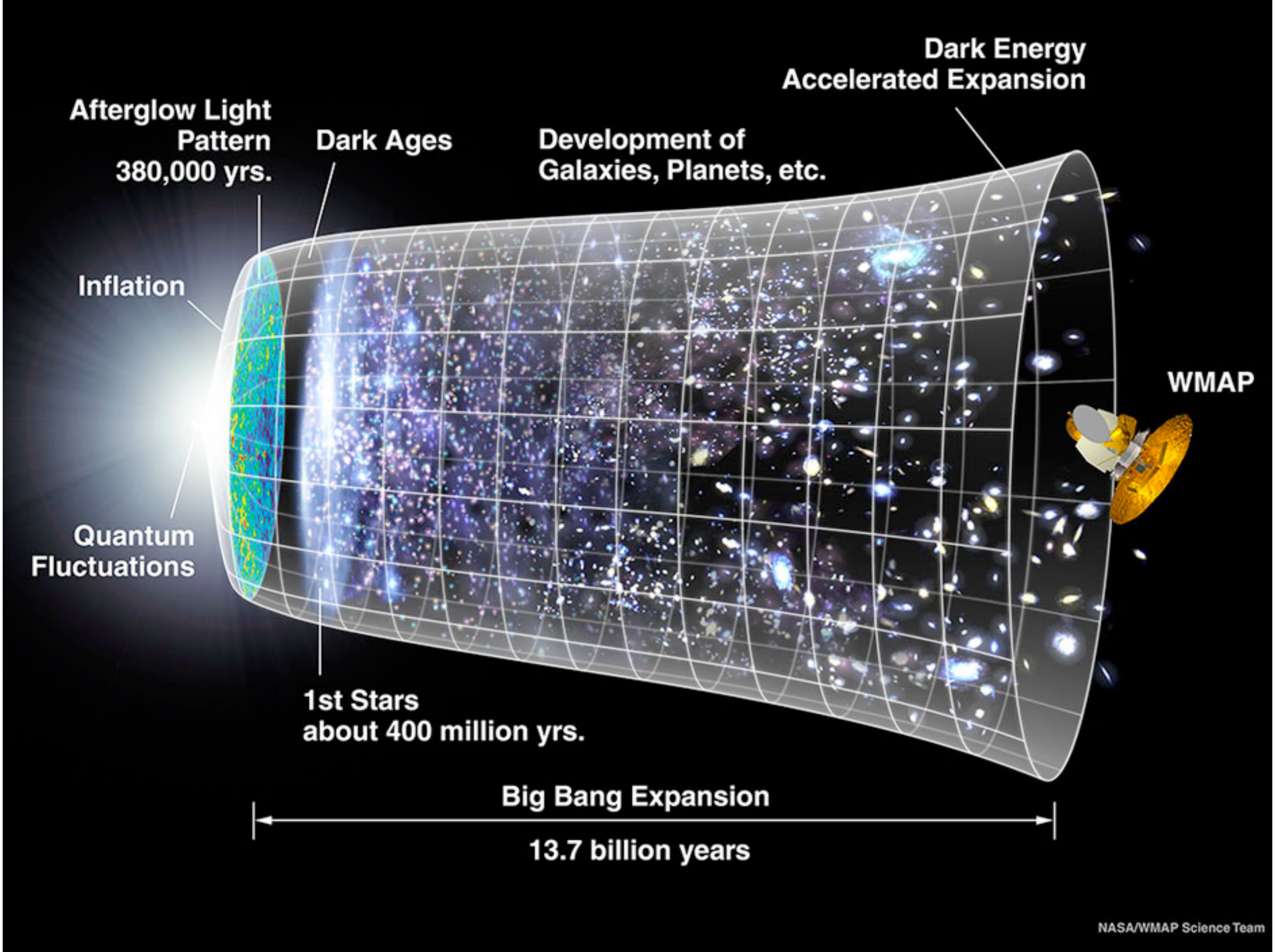
transfer functions: it contains the microphysics - Boltzmann & GR

metric fluctuations as it emerges from inflation



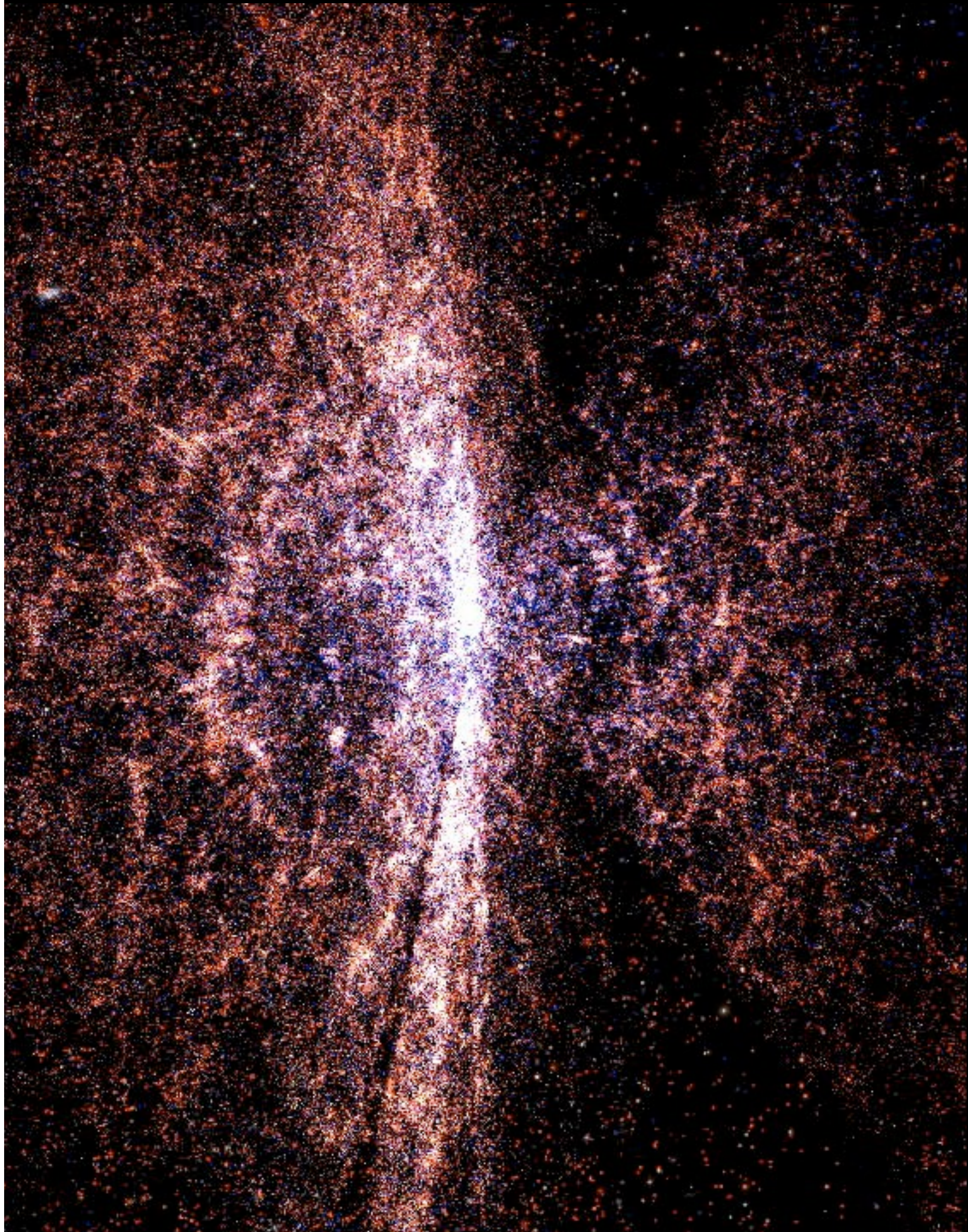
$$ \langle {}_s a_{\ell m}\ {}_s a_{\ell' m'} \rangle = \delta_{\ell\ell'} \delta_{mm'}\ {}_s C_\ell $$

CMB angular power spectra determinations as of mid-2015 (from Planck Collaboration et al. 2015 and Calabrese 2016). This corresponds to the determination (with S/N > 1) of 1 114 000 modes measured with *TT*, 96 000 with *EE* (60 000 with *TE*, not shown), and tens of modes in *BB* (and weak constraints on *T B* and *E B*) .

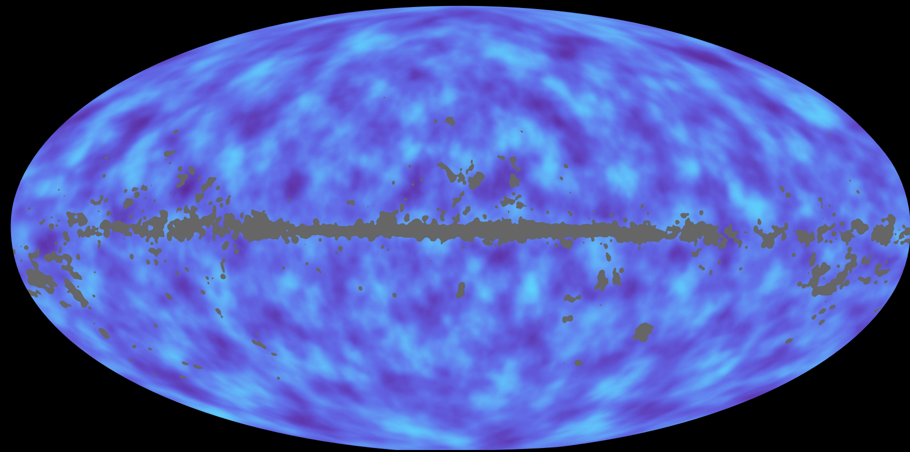


Dark Energy Accelerated Expansion
Afterglow Light Pattern 380,000 yrs.
Dark Ages
Development of Galaxies, Planets, etc.
Inflation
WMAP
Quantum Fluctuations
1st Stars about 400 million yrs.
Big Bang Expansion
13.7 billion years
NASA/WMAP Science Team

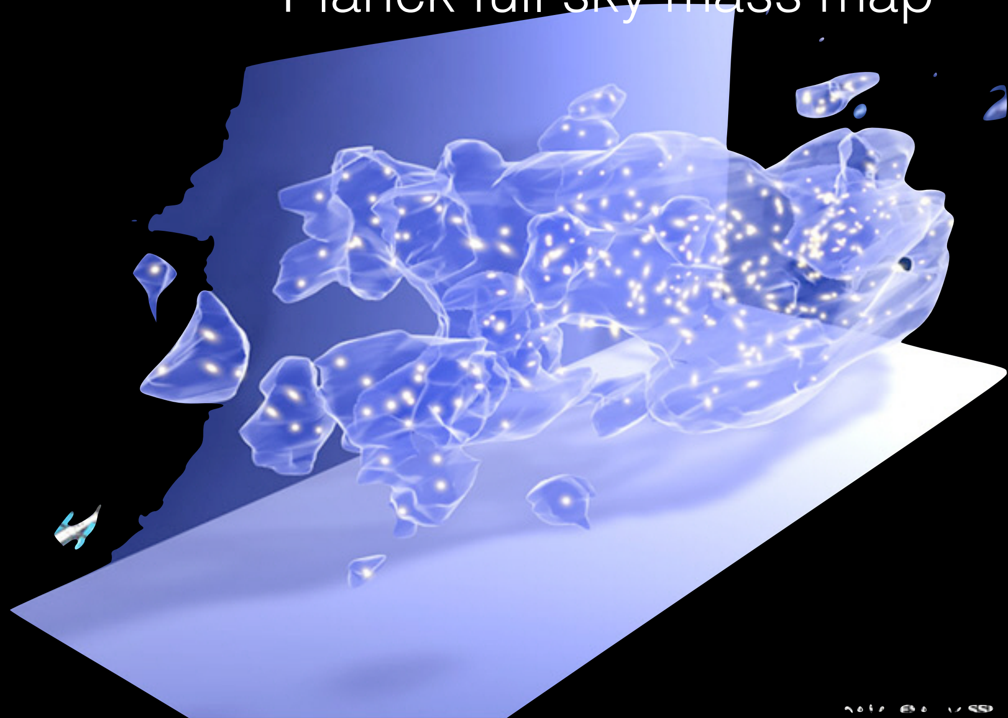
LSS is a (new) treasure trove : many more modes



Galaxy positions in z-space

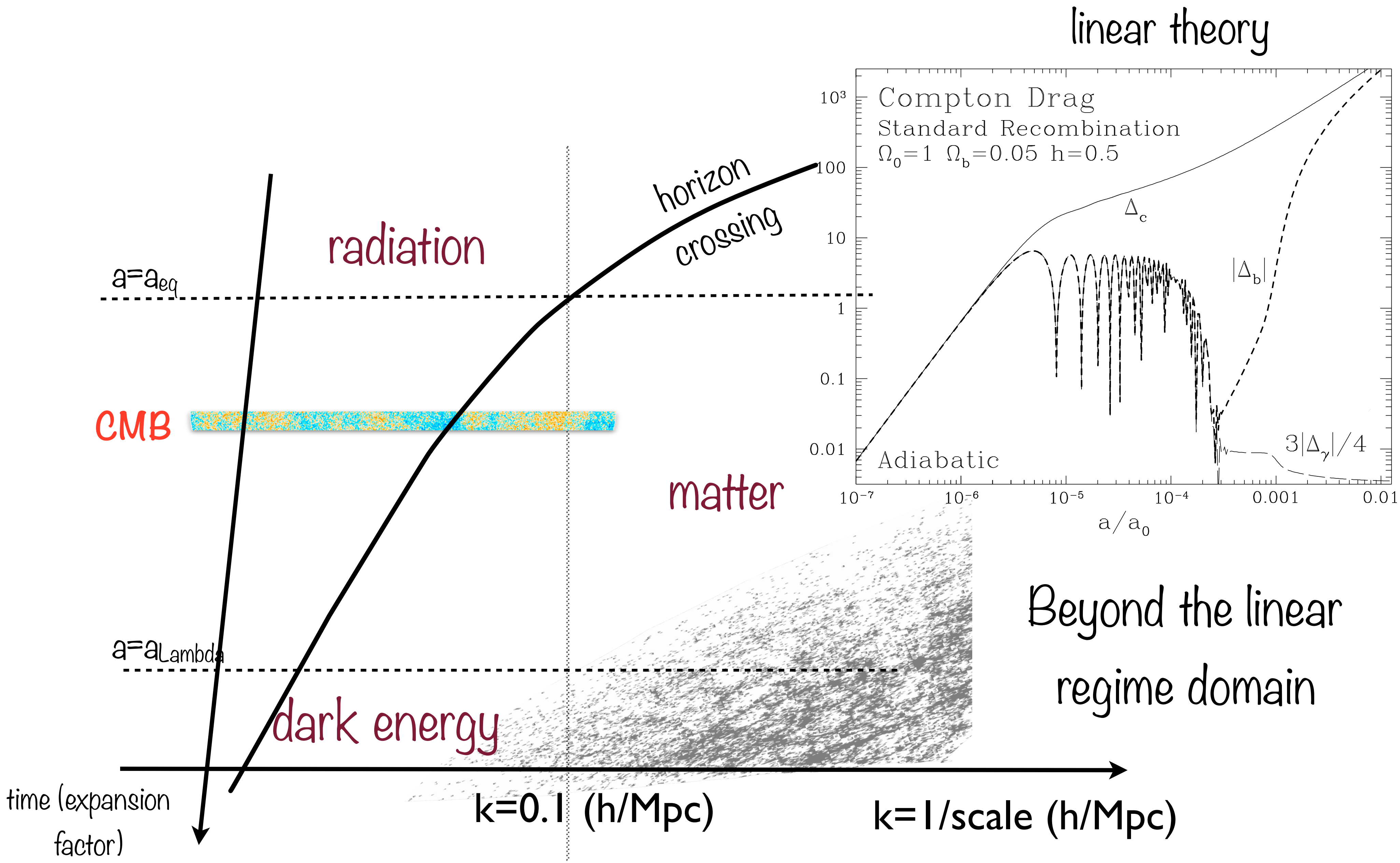


Planck full sky mass map



multi-plane weak lensing
observations

The development of cosmological instabilities across time and scale



Entering the nonlinear regime

Modes are now full functionals of the initial modes. At best they can expended with respect to initial metric fluctuations

$$\delta_\rho(\vec{k}) = \int \mathrm{d}^3\vec{q}\,\mathcal{T}(\vec{k},\vec{q})\,\phi_{\mathrm{adiab}}(\vec{q}) + \int \mathrm{d}^3\vec{q}_1\mathrm{d}^3\vec{q}_2\,\mathcal{T}(\vec{k},\vec{q}_1,\vec{q}_2)\,\phi_{\mathrm{adiab}}(\vec{q}_1)\phi_{\mathrm{adiab}}(\vec{q}_2) + \ldots$$

A theorist work program

- **identify relevant observables (power spectra and beyond)**
- **controlled predictions of identified observables**
- controlled predictions of covariances between such observables (modes are not independent)
- *theoretical* error structure.

A self-gravitating expanding dust fluid

The Vlasov equation

So let us start assuming that the universe is full of dust like particles with the same mass, m.
The first step of the calculation is to introduce the phase space density function, $f(\mathbf{x}, \mathbf{p})\, \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\mathbf{p}$, which is the number of particles per volume element $\mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\mathbf{p}$ where the position $\mathbf{x}$ of the particles is expressed in comoving coordinates and the particle conjugate momentum $\mathbf{p}$ reads

$$\mathbf{p} = \mathbf{u}\, m\, a, \tag{1}$$

where a is the expansion factor, $\mathbf{u}$ is the peculiar velocity, i.e. the difference of the physical velocity of the Hubble expansion.

The Vlasov equation

Then the conservation of the particles together with the Liouville theorem when the the two-body interactions can be neglected implies that the total time derivative of f vanishes so that

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial}{\partial t} f(\mathbf{x}, \mathbf{p}, t) + \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \frac{\partial}{\partial \mathbf{x}} f(\mathbf{x}, \mathbf{p}, t) + \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \frac{\partial}{\partial \mathbf{p}} f(\mathbf{x}, \mathbf{p}, t) = 0. \quad (2)$$

This is the Vlasov equation.

The Vlasov equation

The time variation of the position can be expressed in terms of $\mathbf{p}$ and one gets

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{\mathbf{p}}{m\, a^2}. \tag{3}$$

The time variation of the momentum in general can be obtained from the geodesic equation. Assuming the metric perturbations are small and for scales much below the Hubble scale we have

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -m\nabla_{\mathbf{x}}\Phi(\mathbf{x}, t) \tag{4}$$

where Φ is the potential. We recall that in the context of metric perturbation in an expanding universe the potential $\Phi(\mathbf{x})$ is sourced by the density contrast (of all species). In our context we simply have

$$\Delta\Phi(\mathbf{x}) = \frac{4\pi Gm}{a}\left(\int f(\mathbf{x}, \mathbf{p}, t)\mathrm{d}^3\mathbf{p} - \overline{n}\right) \tag{5}$$

where $\overline{n}$ is the spatial average of $\int f(\mathbf{x}, \mathbf{p}, t)\mathrm{d}^3\mathbf{p}$.

The Vlasov equation

We then have

$$\frac{\partial}{\partial t} f(\mathbf{x}, \mathbf{p}, t) + \frac{\mathbf{p}}{m\, a^2} \frac{\partial}{\partial \mathbf{x}} f(\mathbf{x}, \mathbf{p}, t) - m \nabla_{\mathbf{x}} \Phi(\mathbf{x}, t) \frac{\partial}{\partial \mathbf{p}} f(\mathbf{x}, \mathbf{p}, t) = 0. \quad (6)$$

The system (6, 5) forms the *Vlasov-Poisson* equation. This is precisely the set of equations the N-body simulations attempt to solve.

The motion equations

We can now derive the basic conservation equations we are going to use from the first 2 moments of the Vlasov equation. Let us define the density field per volume $\mathrm{d}^3\mathbf{r}$ as

$$\rho(\mathbf{x}, t) = \frac{m}{a^3} \int \mathrm{d}^3\mathbf{p}\, f(\mathbf{x}, \mathbf{p}). \tag{7}$$

It can be decomposed in an homogeneous form and an inhomogeneous form,

$$\rho(\mathbf{x}, t) = \overline{\rho}(t)(1 + \delta(\mathbf{x}, t)). \tag{8}$$

Note that $\overline{\rho}(t)$ the spatial averaged of $\rho(\mathbf{x}, t)$ should behave like $a(t)^{-3}$ for non relativistic species.

The motion equations

One should then define the higher order moment of the phase space distribution: the mean velocity flow is defined as (for each component),

$$u_i(\mathbf{x}, t) = \frac{1}{\int \mathrm{d}^3\mathbf{p}\, f(\mathbf{x}, \mathbf{p}, t)} \int \mathrm{d}^3\mathbf{p}\, \frac{p_i}{ma} f(\mathbf{x}, \mathbf{p}, t), \tag{9}$$

and the second moment defines the velocity dispersion $\sigma_{ij}(\mathbf{x}, t)$,

$$u_i(\mathbf{x}, t) u_j(\mathbf{x}, t) + \sigma_{ij}(\mathbf{x}, t) = \frac{1}{\int \mathrm{d}^3\mathbf{p}\, f(\mathbf{x}, \mathbf{p}, t)} \int \mathrm{d}^3\mathbf{p}\, \frac{p_i}{ma} \frac{p_j}{ma} f(\mathbf{x}, \mathbf{p}, t). \tag{10}$$

The motion equations

The first two moments of the Vlasov equation give then the conservation and Euler equations, respectively

$$\frac{\partial\delta(\mathbf{x},t)}{\partial t} + \frac{1}{a}\left[(1+\delta(\mathbf{x},t))\mathbf{u}_i(\mathbf{x},t)\right]_{,i} = 0 \tag{11}$$

and

$$\frac{\partial \mathbf{u}_i(\mathbf{x},t)}{\partial t} + \frac{\dot{a}}{a}\mathbf{u}_i(\mathbf{x},t) + \frac{1}{a}\mathbf{u}_j(\mathbf{x},t)\,\mathbf{u}_i(\mathbf{x},t)_{,j} = -\frac{1}{a}\Phi(\mathbf{x},t)_{,i} - \frac{\left(\rho(\mathbf{x},t)\,\sigma_{ij}(\mathbf{x},t)\right)_{,j}}{\rho(\mathbf{x},t)\,a}. \tag{12}$$

The first term of the right hand side of eqn (12) is the gravitational force, the second is due to the pressure force which in general can be anisotropic. In the context we are interested in, it actually vanishes until the formation of the first caustics.

Single flow approximation

The early stages of the gravitational instabilities are indeed characterized, assuming the matter is non-relativistic, by a negligible velocity dispersion when it is compared to the velocity flows, i.e. much smaller than the velocity gradients induced by the density fluctuations of the scales of interest. This is the *single flow approximation*. It simply states that one can assume

$$f(\mathbf{x},\mathbf{p},t)=\frac{a^3\,\rho(\mathbf{x},t)}{m}\,\delta^{(3)}[\mathbf{p}-m\,a\,\mathbf{u}(\mathbf{x},t)], \tag{13}$$

to a good approximation. This approximation will naturally break at the time of shell crossings when different flows – pulled toward one-another by gravity – cross. The Vlasov-Poisson equation in the single flow regime is the system that will be studied throughout these lecture notes, from linear to non nonlinear regime.

Single flow approximation

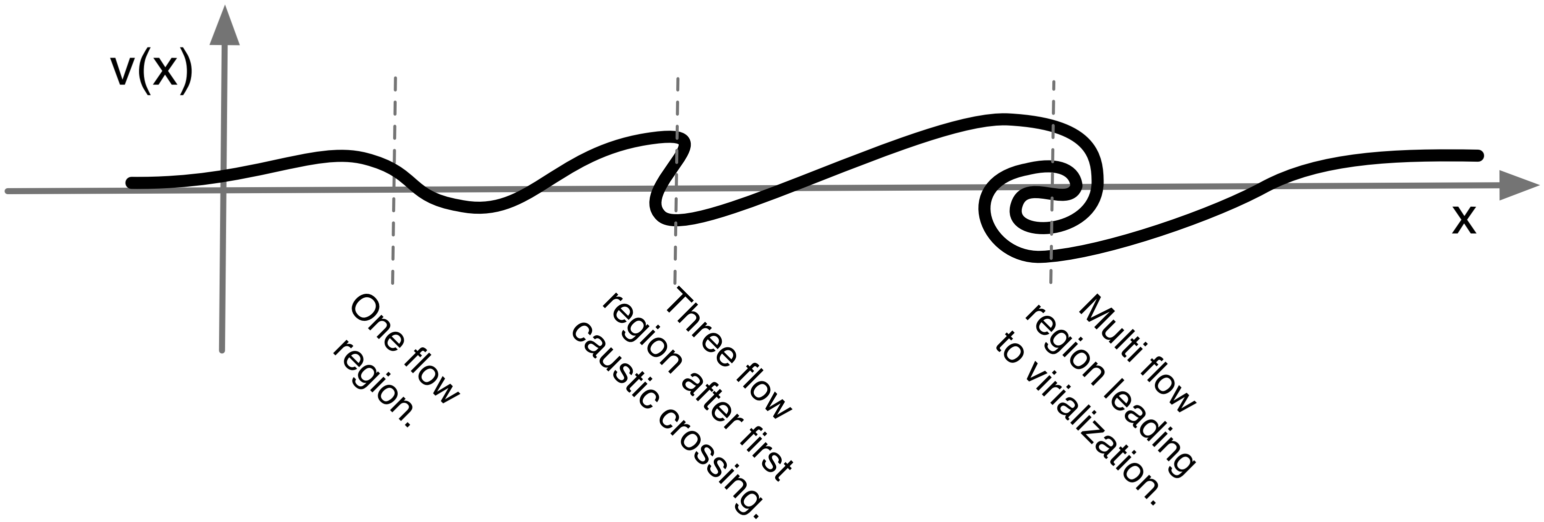


Figure: Schematic description of phase space after the first shell crossings and emergence of multi-flow regions. The figure is for 1D dynamics. From left to right, one can see regions with growing number of flows after dark matter caustic crossings.

The curl modes

In the single flow regime, one can note that the source term of the Euler equation is potential, implying that it cannot generate any curl mode in the velocity field. More precisely, one can decompose any three-dimensional field in a gradient part and a curl part

$$\mathbf{u}_i(\mathbf{x}) = \psi(\mathbf{x})_{,i} + \mathbf{w}_i(\mathbf{x}) \tag{14}$$

where $\mathbf{w}_{i,i} = 0$. Defining the local vorticity as

$$\omega_k(\mathbf{x}) = \epsilon^{ijk}\mathbf{u}_{i,j}(\mathbf{x}) \tag{15}$$

where ϵ^{ijk} is the totally anti-symmetric Levi-Civita tensor one can easily show that

$$\omega_k(\mathbf{x}) = \epsilon^{ijk}\mathbf{w}_{i,j}(\mathbf{x}). \tag{16}$$

The curl modes

And applying the operator $\epsilon^{ijk}\nabla_j$ to the Euler equation one gets (see [?])

$$\frac{\partial}{\partial t}\omega_k + \frac{\dot{a}}{a}\omega_k - \epsilon^{ijk}\epsilon^{lmi}\left(\mathbf{u}_l\,\omega_m\right)_{,j} = 0. \tag{17}$$

This equation actually expresses the fact that the vorticity is conserved throughout the expansion. In the linear regime that is when the last term of this equation is dropped it simply means that the vorticity scales like $1/a$. In the subsequent stage of the dynamics the vorticity can only grow in contracting regions but it is still somehow conserved, it cannot be created out of potential modes only. That will be the case until shell crossing where the anisotropic velocity dispersion can then induce vorticity. This has been explicitly demonstrated in various studies [**?**, **?**, **?**]. As a consequence, in the following, curl modes in the vector flied will always be neglected.

The linear theory

We now proceed to explore the linear regime of the Vlasov-Poisson system. One objective is to make contact with earlier stages of the gravitational dynamics and the second is to introduce the notion of Green function we will use in the following.

The linear modes

The linearization of the motion equation is obtained when one assumes that the terms $[\delta(\mathbf{x},t))\mathbf{u}_i(\mathbf{x},t)]_{,i}$ and $\mathbf{u}_j(\mathbf{x},t)\,\mathbf{u}_i(\mathbf{x},t)_{,j}$ in respectively the continuity and the Euler equation vanish. This is obtained when both the density contrast and the velocity gradients in units of H are negligible. The linearized system is obtained in terms of the velocity divergence

$$\theta(\mathbf{x},t) = \frac{1}{aH} u_{i,i} \tag{18}$$

so that the system now reads

$$\begin{aligned} \frac{\partial}{\partial t}\delta(\mathbf{x},t) + H\theta(\mathbf{x},t) &= 0 && (19)\\ \frac{\partial}{\partial t}\theta(\mathbf{x},t) + 2H\theta + \frac{\dot{H}}{H}\theta(\mathbf{x},t) &= -\frac{3}{2}H\Omega_m(t)\delta(\mathbf{x},t) && (20) \end{aligned}$$

after taking the divergence of the Euler equation. We have introduced here the Hubble parameter $H = \dot{a}/a$ and used the Friedman equation $H^2 = 8\pi/3\ G\rho_c(t)$ together with the definition of $\Omega_m = \overline{\rho}(t)/\rho_c(t)$.

The linear modes

The resolution of this system is now simple. It can be obtained after eliminating the velocity divergence and one gets a second order dynamical equation,

$$\frac{\partial^2}{\partial t^2}\delta(\mathbf{x},t) + 2\,H\frac{\partial}{\partial t}\delta(\mathbf{x},t) - \frac{3}{2}\,H^2\,\Omega_m\,\delta(\mathbf{x},t) = 0, \qquad (21)$$

for the density contrast. It is to be noted that the spatial coordinates are here just labels: there is no operator acting of the physical coordinates. This is quite a unique feature in the growth of instabilities in a pressureless fluid. That implies in particular that the linear growth rate of the fluctuations will be independent on scale. The time dependence of the linear solution is given by the two solutions of

$$\ddot{D} + 2\,H\,\dot{D} - \frac{3}{2}\,H^2\,\Omega_m\,D = 0, \qquad (22)$$

one of which is decaying and the other is growing with time.

The linear modes, Einstein de Sitter case

For an Einstein de-Sitter (EdS) background (a universe with no curvature and with a critical matter density) the solutions read

$$D_+^{\rm EdS}(t) \propto t^{2/3}, \quad D_-^{\rm EdS}(t) \propto 1/t, \tag{23}$$

that is $D_+^{\rm EdS}(t)$ is proportional to the expansion factor. This result gives the time scale of the growth of structure. This is what permits a direct comparison between the amplitude of the metric perturbations at recombination and the density perturbation in the local universe. Note that it implies that the potential, for the corresponding mode, is constant (see the Poisson equation).

The linear modes, general solutions

Il n'existe pas de solution générale à (22) qui soit valable pour tout modèle cosmologique. Cependant si on admet que le contenu de l'univers à bas redshift correspond à un mélange de matière, d'une énergie du vide correspondant à une simple constante cosmologique, en présence éventuellement d'un terme de courbure, alors la constante de Hubble prend la forme,

$$H = H_0 \sqrt{\Omega_{\rm m}^{(0)} a^{-3} + (1 - \Omega_{\rm m}^{(0)} - \Omega_{\sim}^{(0)}) a^{-2} + \Omega_\Lambda}, \qquad (24)$$

et est solution de (22). C'est la solution décroissante. La solution croissante peut facilement être obtenue par méthode de variation des constantes. On obtient ainsi, sous les hypothèses mentionnées plus haut,

$$D_-(t) \propto H(t), \qquad (25)$$

$$D_+(t) \propto H(t) \int \frac{\mathrm{d}t}{(aH)^2}. \qquad (26)$$

The linear theory

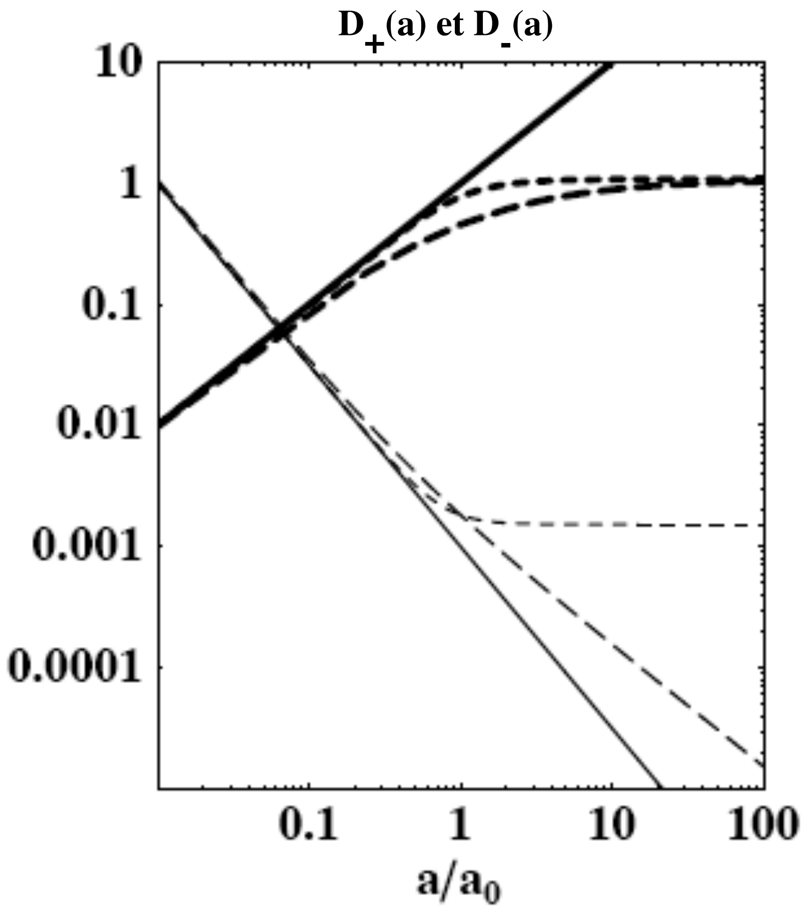


Figure: Comportement des taux de croissance (lignes épaisses) et de décroissance (lignes fines) des instabilités gravitationnelles pour un univers Einstein-de Sitter (lignes continues), un univers avec $\Omega_{\rm m}^{(0)} = 0.3,\ \Omega_\Lambda = 0$ (lignes pointillées longues) et un univers avec $\Omega_{\rm m}^{(0)} = 0.3,\ \Omega_\Lambda^{(0)} = 0.7$ (lignes pointillées courtes).

The linear theory

Si on se restreint à des cosmologies sans constante cosmologique, $\Omega_\Lambda = 0$, D_+ peut s'exprimer analytiquement en fonction de a,

$$D_+(a) \propto 1 + \frac{3\Omega_{\rm m}^{(0)}}{a - a\Omega_{\rm m}^{(0)}} + 3\sqrt{\frac{(a(\Omega_{\rm m}^{(0)} - 1) - \Omega_{\rm m}^{(0)})\Omega_{\rm m}^{(0)\,2}}{a^3(\Omega_{\rm m}^{(0)} - 1)^3}} \times \log\left(\sqrt{a\left(\frac{1}{\Omega_{\rm m}^{(0)}} - 1\right) + 1} - \sqrt{a\left(\frac{1}{\Omega_{\rm m}^{(0)}} - 1\right)}\right)$$

The linear theory

Pour un contenu en énergie matière arbitraire, il faut résoudre bien sûr l'équation (22). Les résultats d'une telle intégration sont montrés sur la figure 2. Une forme plus simple existe cependant dans le cas d'univers plat avec constante cosmologique. Dans ce cas la dépendance temporelle de D_+ est donnée par

$$D_+(t) = {}_2F_1\left(1, \frac{1}{3}; \frac{11}{6}; -\sinh^2\left(\frac{3\alpha t}{2}\right)\right)\sinh^{\frac{2}{3}}\left(\frac{3\alpha t}{2}\right), \qquad (27)$$

et le facteur f prend la forme

$$f = 1 - \frac{6}{11}\frac{{}_2F_1\left(2, \frac{4}{3}; \frac{17}{6}; -\sinh^2\left(\frac{3\alpha t}{2}\right)\right)\sinh^2\left(\frac{3\alpha t}{2}\right)}{{}_2F_1\left(1, \frac{1}{3}; \frac{11}{6}; -\sinh^2\left(\frac{3\alpha t}{2}\right)\right)} \qquad (28)$$

avec $\alpha = \sqrt{\Lambda/3}$.

La relation densité-vitesse

Pour être complet, il faut bien sûr expliciter la relation entre le contraste de densité et la divergence du champ de vitesse. L'équation de continuité peut se réécrire,

$$a\frac{\partial}{\partial a}\delta + \theta = 0. \tag{29}$$

Le champ θ s'écrit donc,

$$\theta(t, \mathbf{x}) = \frac{\partial \log D_+}{\partial \log a}\delta_+(t, \mathbf{x}) + \frac{\partial \log D_-}{\partial \log a}\delta_-(t, \mathbf{x}) \tag{30}$$

où $\delta_\pm(t, \mathbf{x}) = D_\pm(t)\delta_\pm(\mathbf{x})$. Pour un univers Einstein-de Sitter, cette relation devient

$$\theta(t, \mathbf{x}) = \delta_+(t, \mathbf{x}) - \frac{3}{2}\delta_-(t, \mathbf{x}). \tag{31}$$

La relation densité-vitesse

Poursuivons l'examen de la relation densité-vitesse. Si on se limite au terme croissant le facteur de proportionnalité entre θ et δ est donc la dérivée logarithmique de D_+ avec le facteur d'expansion,

$$f \equiv \frac{\mathrm{d} \log D_+}{\mathrm{d} \log a}. \tag{34}$$

Ce facteur vaut 1 pour un univers Einstein de Sitter. On peut bien sûr le calculer analytiquement ou numériquement selon les cas. Par exemple pour un univers plat avec constante cosmologique une bonne paramétrisation de f est,

$$f(\Omega_{\mathrm{m}}) = \Omega^{5/9} \approx \Omega^{0.55}; \tag{35}$$

et pour un univers sans constante cosmologique,

$$f(\Omega_{\mathrm{m}}) = \Omega^{3/5}. \tag{36}$$

La figure (4) compare les résultats exacts à ces formes approchées. On voit que l'écart est tout au plus de quelques pour-cents.

La relation densité-vitesse

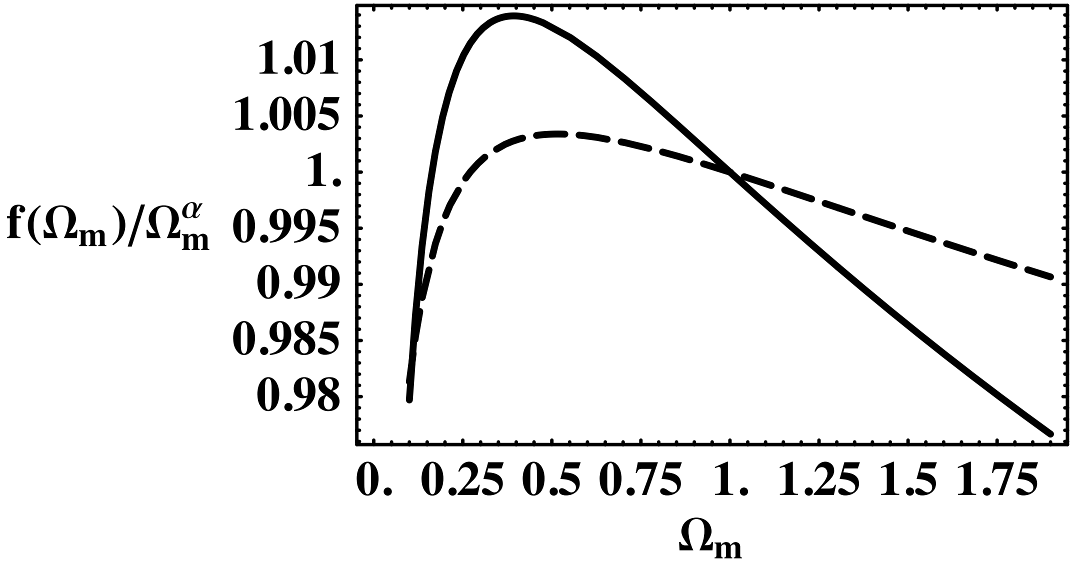


Figure: Facteur f, équation (34), divisé par sa valeur estimée, équations (35-36), en fonction de Ω_m pour un espace sans constante cosmologique (ligne continue) ou pour un espace plat (tirets).

Linear theory: the Green functions

The previous results show that the linear density field can be written in general

$$\delta(\mathbf{x}, t) = \delta_+(\mathbf{x}) D_+(t) + \delta_-(\mathbf{x}) D_-(t) \tag{37}$$

and

$$\theta(\mathbf{x}, t) = -\frac{\mathrm{d}}{\mathrm{d}\log a} D_+ \; \delta_+(\mathbf{x}) - \frac{\mathrm{d}}{\mathrm{d}\log a} D_- \delta_-(\mathbf{x}). \tag{38}$$

The actual growing and decaying modes can then be obtained by inverting this system. For instance for an Einstein de Sitter background one gets

$$\delta_+(\mathbf{x}) D_+(t) = \frac{D_+(t)}{D_+(t_0)} \left[\frac{3}{5}\delta(\mathbf{x}, t_0) - \frac{2}{5}\theta(\mathbf{x}, t_0)\right], \tag{39}$$

$$\delta_-(\mathbf{x}) D_-(t) = \frac{D_-(t)}{D_-(t_0)} \left[\frac{2}{5}\delta(\mathbf{x}, t_0) + \frac{2}{5}\theta(\mathbf{x}, t_0)\right], \tag{40}$$

and similar results for the velocity divergence.

Linear theory: the Green functions

Following [?], this result can be encapsulated in a simple form after one introduces the doublet $\Psi_a(\mathbf{k},\tau)$,

$$\Psi_a(\mathbf{k},\tau) \equiv \Big(\delta(\mathbf{x},t),\ -\theta(\mathbf{x},t)\Big), \tag{41}$$

where a is an index whose value is either 1 (for the density component) or 2 (for the velocity component).

Linear theory: the Green functions

The linear growth solution can now be written

$$\Psi_a(\mathbf{x}, t) = g_a{}^b(t, t_0)\Psi_b(\mathbf{x}, t_0) \tag{42}$$

where g_a^b is the Green function of the system. It is usually written with the following time variable,

$$\eta = \log D_+ \tag{43}$$

(not to be mistaken with the conformal time). For an Einstein de Sitter universe, we have explicitly,

$$g_a{}^b(\eta, \eta_0) = \frac{e^{\eta-\eta_0}}{5}\begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} + \frac{e^{-\frac{3}{2}(\eta-\eta_0)}}{5}\begin{bmatrix} 2 & -2 \\ -3 & 3 \end{bmatrix}. \tag{44}$$

We will see in the following that, provided the doublet Ψ_a is properly defined, this form remains practically unchanged for any background.

Linear theory: The general background case

For a general background, it is fruitful to extent the definition of the doublet to,

$$\Psi_a(\mathbf{x},\eta) \equiv \left(\delta(\mathbf{x},\eta),\ -\frac{1}{f_+}\theta(\mathbf{x},\eta)\right), \tag{45}$$

where

$$f_+ = \frac{\mathrm{d}\log D_+}{\mathrm{d}\log a} \tag{46}$$

Defining $\hat{\theta} = -\theta(\mathbf{x},\eta)/f_+$ and for the time variable η, the linearized motion equations indeed read

$$\frac{\partial}{\partial\eta}\delta(\mathbf{x},\eta) - \hat{\theta}(\mathbf{x},\eta) = 0 \tag{47}$$

$$\frac{\partial}{\partial\eta}\hat{\theta}(\mathbf{x},\eta)\left(\frac{3}{2}\frac{\Omega_m}{f_+^2} - 1\right)\hat{\theta} - \frac{3}{2}\frac{\Omega_m}{f_+^2}\delta(\mathbf{x},\eta) = 0, \tag{48}$$

Cosmic fields as statistical objects

The origin of stochasticity

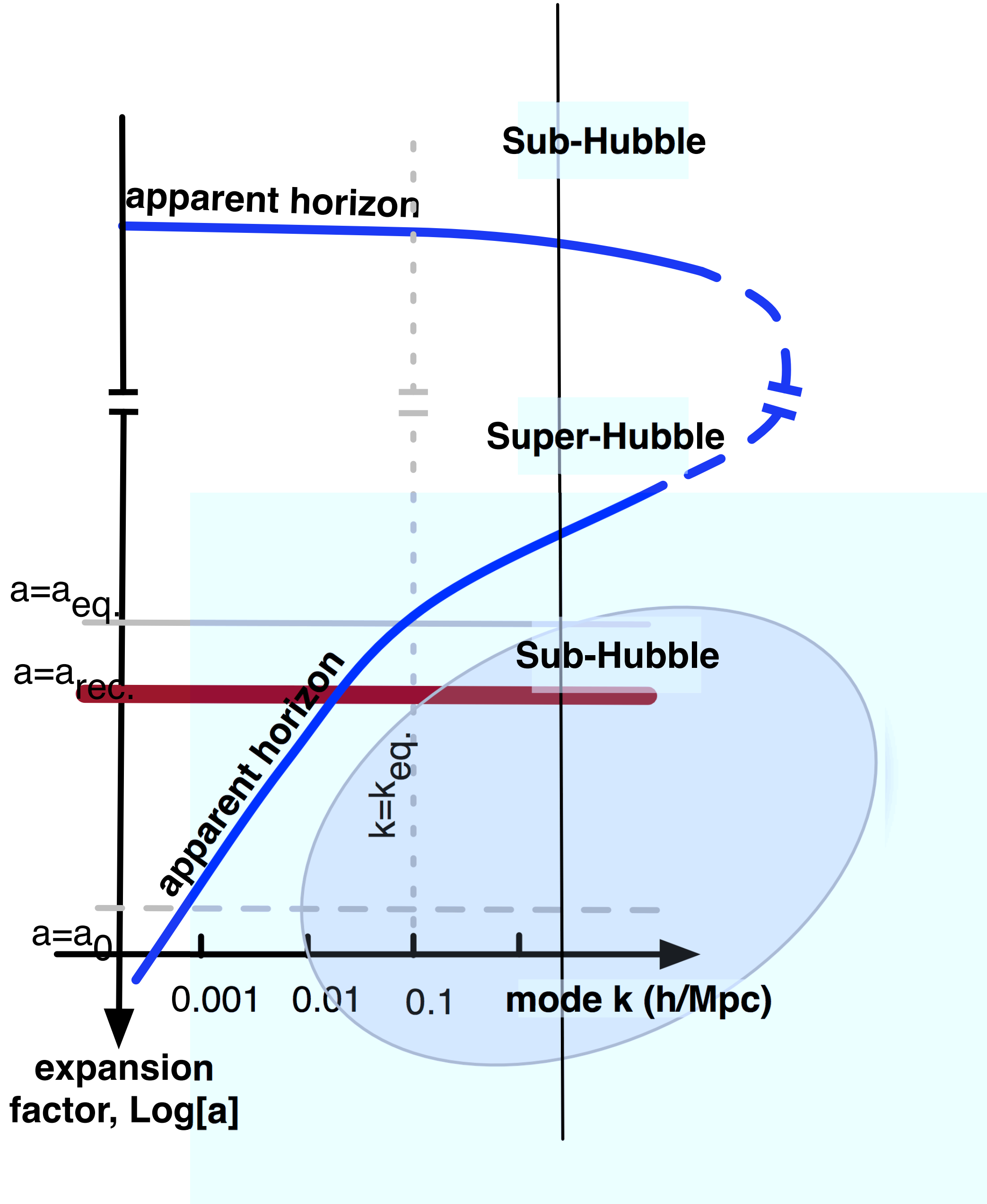


Figure: Development of linear metric perturbation across scale and time.

The origin of stochasticity

In models of inflation the stochastic properties of the fields originate from quantum fluctuations of a scalar field, the inflaton. This field has quantum fluctuations that can be decomposed in Fourier modes using the creation and annihilation operators $a_{\mathbf{k}}^{\dagger}$ and $a_{\mathbf{k}}$ for a wave mode $\mathbf{k}$,

$$\delta\varphi = \int \mathrm{d}^3\mathbf{k} \left[a_{\mathbf{k}}\, \psi_k(t)\, \exp(\mathrm{i}\mathbf{k}.\mathbf{x}) + a_{\mathbf{k}}^{\dagger}\, \psi_k^*(t)\, \exp(-\mathrm{i}\mathbf{k}.\mathbf{x}) \right]. \quad (55)$$

The operators obey the standard commutation relation,

$$[a_{\mathbf{k}}, a_{-\mathbf{k}'}^{\dagger}] = \delta_D(\mathbf{k} + \mathbf{k}'), \quad (56)$$

and the mode functions $\psi_k(t)$ are obtained from the Klein-Gordon equation for φ in an expanding Universe.

The origin of stochasticity

We give here its expression for a de-Sitter metric (i.e. when the spatial sections are flat and H is constant),

$$\psi_k(t) = \frac{H}{(2\,k)^{1/2}\,k}\left(\mathrm{i} + \frac{k}{a\,H}\right)\,\exp\left[\frac{\mathrm{i}\,k}{a\,H}\right], \qquad (57)$$

where a and H are respectively the expansion factor and the Hubble constant that are determined by the overall content of the Universe through the Friedmann equations.
When the modes exit the Hubble radius, $k/(aH) \ll 1$, one can see from eqn (57) that the dominant mode reads,

$$\varphi_{\mathbf{k}} \approx \frac{iH}{\sqrt{2}k^{3/2}}\left(a_{\mathbf{k}} + a^{\dagger}_{-\mathbf{k}}\right), \quad \delta\varphi = \int \mathrm{d}^3\mathbf{k}\,\varphi_{\mathbf{k}}\,e^{\mathrm{i}\,\mathbf{k}.\mathbf{x}}. \qquad (58)$$

Therefore these modes are all proportional to $a_{\mathbf{k}} + a^{\dagger}_{-\mathbf{k}}$.

The origin of stochasticity

One important consequence of this is that the quantum nature of the fluctuations has disappeared [**?**, **?**]: any combinations of $\varphi_{\mathbf{k}}$ commute with each other. The field φ can then be seen as a classic stochastic field where *ensemble averages identify with vacuum expectation values*,

$$\langle ... \rangle \equiv \langle 0 | ... | 0 \rangle. \tag{59}$$

The origin of stochasticity

After the inflationary phase the modes re-enter the Hubble radius. They leave imprints of their energy fluctuations in the gravitational potential, the statistical properties of which can therefore be deduced from eqns (56, 58). All subsequent stochasticity that appears in the cosmic fields can thus be expressed in terms of the random variable $\varphi_{\mathbf{k}}$. The linear theory calculation precisely tells us how each mode, in each fluid component, grows across time, i.e. it provides us with the so-called transfer functions, $T_a(\mathbf{k}, \eta, \eta_0)$, defined as

$$\delta_a(\mathbf{k}, \eta) = T_a(\mathbf{k}, \eta, \eta_0)\delta\varphi(\mathbf{k}, \eta_0) \tag{60}$$

where η_0 is a time which corresponds to an arbitrarily early time.

Statistical homogeneity and isotropy

In the following the density contrast will be decomposed in Fourier modes that, for a flat universe, are defined such as

$$\delta(\mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}\,\delta(\mathbf{k})\,\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x}) \tag{61}$$

or equivalently

$$\delta(\mathbf{k}) = \int \frac{\mathrm{d}^3\mathbf{x}}{(2\pi)^{3/2}}\,\delta(\mathbf{x})\,\exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{x}). \tag{62}$$

The observable quantities of interest are actually the statistical properties of such a field, whether it is represented in real space or in Fourier space.

Statistical homogeneity and isotropy

The *Cosmological Principle*, e.g. that the assumption that the Universe is statically isotropic and homogeneous, implies that real space correlators are homogeneous and isotropic which for instance implies that $\langle \delta(\mathbf{x})\delta(\mathbf{x}+\mathbf{r})\rangle$ is a function of the separation r only. This defines the two-point correlation function,

$$\xi(r) = \langle \delta(\mathbf{x})\delta(\mathbf{x}+\mathbf{r})\rangle. \tag{63}$$

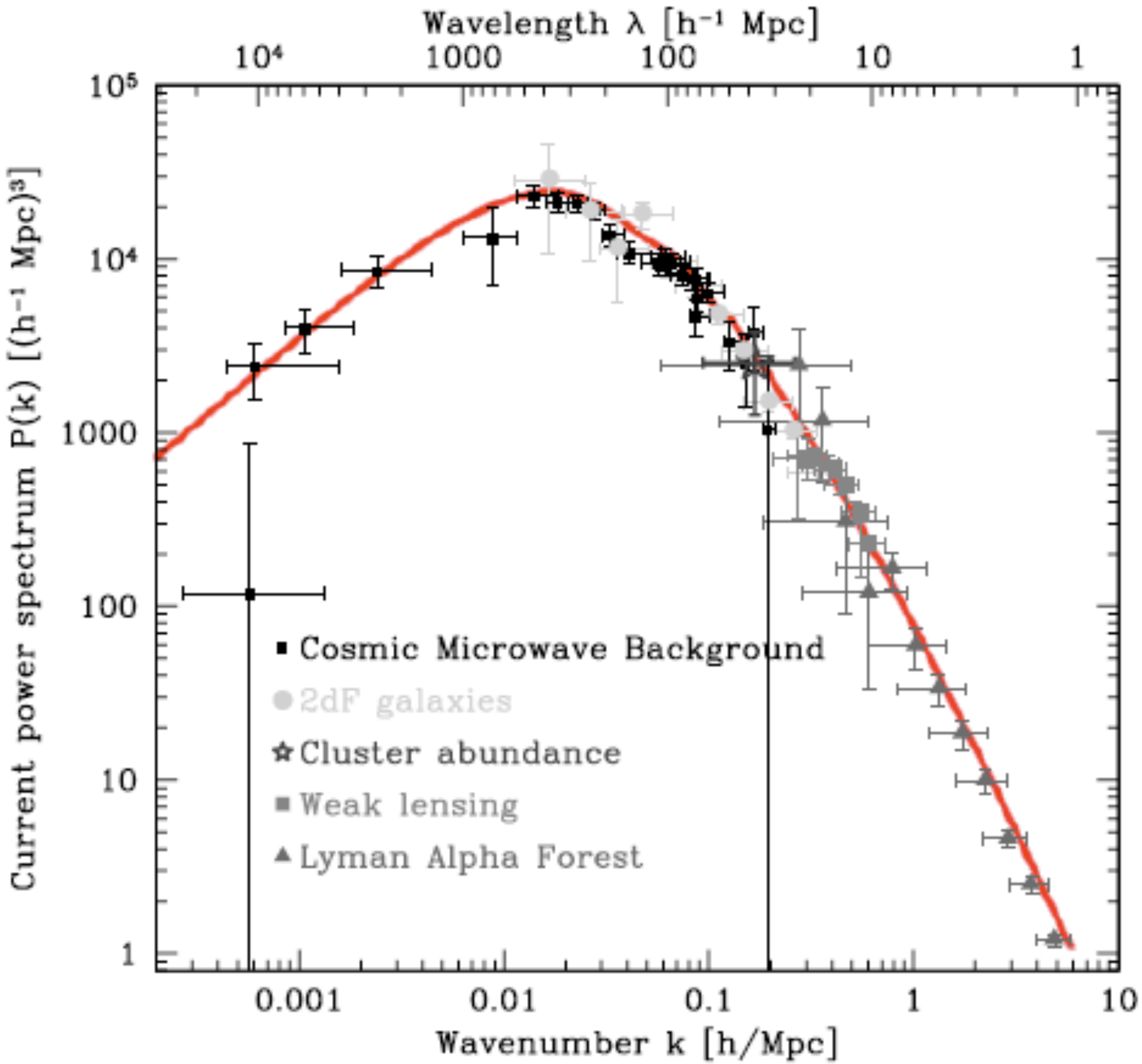
In Fourier space, the two point correlator of the Fourier modes then takes the form,

$$\begin{aligned}
\langle \delta(\mathbf{k})\delta(\mathbf{k}')\rangle &= \int \frac{\mathrm{d}^3\mathbf{x}}{(2\pi)^{3/2}} \frac{\mathrm{d}^3\mathbf{r}}{(2\pi)^{3/2}}\, \xi(r) \exp[-\mathrm{i}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x} - \mathrm{i}\mathbf{k}'\cdot\mathbf{r}] \\
&= \delta_{\mathrm{Dirac}}(\mathbf{k}+\mathbf{k}') \int \mathrm{d}^3\mathbf{r}\, \xi(r) \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}) \\
&\equiv \delta_{\mathrm{Dirac}}(\mathbf{k}+\mathbf{k}')\, P(k),
\end{aligned} \tag{64}$$

where $P(k)$ is the *power spectrum* of the density field, e.g. the cross-correlation matrix is symmetric in Fourier space.

Power spectrum

$$\langle \delta_{\mathbf{k}} \delta_{\mathbf{k}'} \rangle = \delta_{\rm Dirac}(\mathbf{k} + \mathbf{k}')\, P(k)$$



Statistical homogeneity and isotropy

All these relations apply to the observed fields. However in case of the primordial fluctuations, the field $\delta\varphi(\mathbf{k})$ corresponds to a free field from a quantum mechanical point of view. That makes it eventually a Gaussian classical field. As such it obeys the Wick theorem. The latter tells us that higher order correlators can then be entirely constructed from the power spectrum (from pair associations) through the relations,

$$\begin{aligned} \left\langle \delta\varphi(\mathbf{k}_1)\ldots\delta\varphi(\mathbf{k}_{2p+1})\right\rangle &= 0 \\ \left\langle \delta\varphi(\mathbf{k}_1)\ldots\delta\varphi(\mathbf{k}_{2p})\right\rangle &= \sum_{\text{pair associations}} \prod_{p \text{ pairs (i,j)}} \left\langle \delta\varphi(\mathbf{k}_i)\delta\varphi(\mathbf{k}_j)\right\rangle. \end{aligned} \tag{65}$$

These relations apply as well to any linear combinations of the primordial field, and therefore to any field computed in the linear regime.

Moments and cumulants

In the nonlinear regime however, fields also exhibit higher order non-trivial correlation functions that cannot be reconstructed from the two-point order correlators. They are defined as the *connected* part (denoted with subscript c) of the joint ensemble average of fields in an arbitrarily number of locations. Formally, for the density field, it reads,

$$\begin{aligned}\langle \delta(\mathbf{x}_1), \ldots, \delta(\mathbf{x}_N) \rangle &= \langle \delta(\mathbf{x}_1), \ldots, \delta(\mathbf{x}_N) \rangle_c \\ &+ \sum_{\mathcal{S} \in \mathcal{P}(\{\mathbf{x}_1, \ldots, \mathbf{x}_n\})} \prod_{s_i \in \mathcal{S}} \langle \delta(\mathbf{x}_{s_i(1)}), \ldots, \delta(\mathbf{x}_{s_i(\# s_i)}) \rangle_c , \end{aligned} \quad (66)$$

where the sum is made over the proper partitions (any partition except the set itself) of $\{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ and s_i is thus a subset of $\{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ contained in partition $\mathcal{S}$. When the average of $\delta(\mathbf{x})$ is defined as zero, only partitions that contain no singlets contribute.

Moments and cumulants

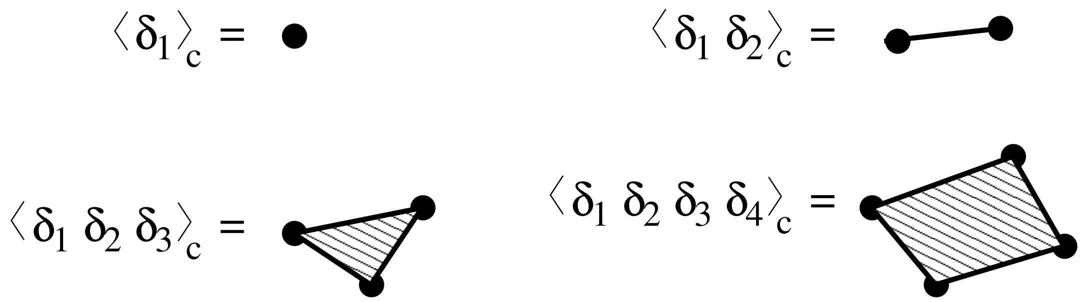


Figure: Representation of the connected part of the moments.

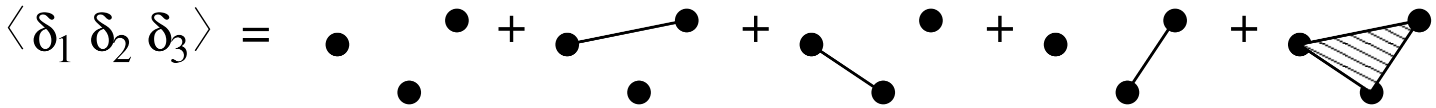


Figure: Writing of the three-point moment in terms of connected parts.

Moments and cumulants

The decomposition in connected and non-connected parts can be easily visualized. It means that any ensemble average can be decomposed in a product of connected parts. They are defined for instance in Fig. 6. The tree-point moment is "written" in Fig. 7. Because of homogeneity of space $\langle\delta(\mathbf{k}_1)\ldots\delta(\mathbf{k}_N)\rangle_c$ is always proportional to $\delta_D(\mathbf{k}_1+\cdots+\mathbf{k}_N)$. Then we can define $P_N(\mathbf{k}_1,\ldots,\mathbf{k}_N)$ with

$$\langle\delta(\mathbf{k}_1)\ldots\delta(\mathbf{k}_N)\rangle_c=\delta_D(\mathbf{k}_1+\cdots+\mathbf{k}_N)\,P_N(\mathbf{k}_1,\ldots,\mathbf{k}_N). \quad (67)$$

One case of particular interest is for $n=3$, the bispectrum, which is usually denoted by $B(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$. Note that it depends on 2 wave modes only, and it depends on 3 independent variables characterizing the triangle formed by the 3 wave modes (for instance 2 lengths and 1 angle).

Moment and cumulant generating functions

It is convenient to define a function from which all moments can be generated, namely the *moment generating function*. It can be defined[2] for any number of random variables. Here we give its definition for the local density field. It is defined by

$$\mathcal{M}(t) \equiv \sum_{p=0}^{\infty} \frac{\langle \rho^p \rangle}{p!} t^p = \langle \exp(t\delta) \rangle. \tag{68}$$

The moments can obviously obtained by subsequent derivatives of this function at the origin $t = 0$. A cumulant generating function can similarly be defined by

$$\mathcal{C}(t) \equiv \sum_{p=2}^{\infty} \frac{\langle \rho^p \rangle_c}{p!} t^p. \tag{69}$$

[2] It is to be noted however that the existence of moments – which itself is not guaranteed for any stochastic process – does not ensure the existence of their generating function as the series defined in (68) can have a vanishing converging radius. Such a case is encountered for a lognormal distribution for instance and it implies that the moments of such a stochastic process do not uniquely define the probability distribution function.

Moment and cumulant generating functions

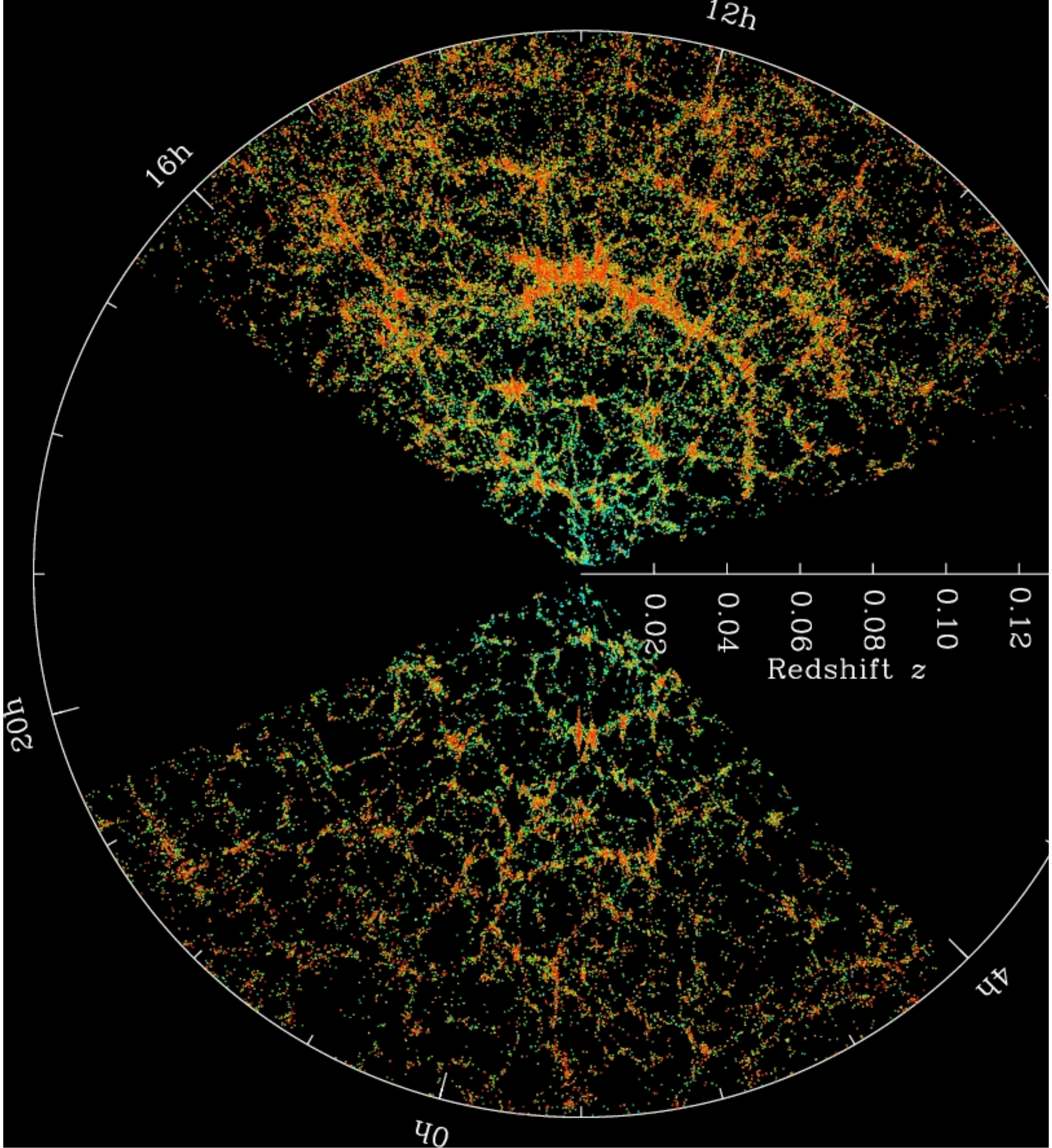
A fundamental result is that the cumulant generating function is given by the logarithm of the moment generation function

$$\mathcal{M}(t) = \exp[\mathcal{C}(t)]. \tag{70}$$

In case of a Gaussian probability distribution function, this is straightforward to check since $\langle \exp(t\delta) \rangle = \exp(\sigma^2 t^2/2)$.

Francis Bernardeau

La structuration des galaxies, théorie



Into the nonlinear universe

Linear theory: The general background case

For a general background, the doublet is defined as,

$$\Psi_a(\mathbf{x},\eta) \equiv \left(\delta(\mathbf{x},\eta),\ -\frac{1}{f_+}\theta(\mathbf{x},\eta)\right), \tag{45}$$

where

$$f_+ = \frac{\mathrm{d}\log D_+}{\mathrm{d}\log a} \tag{46}$$

Defining $\hat{\theta} = -\theta(\mathbf{x},\eta)/f_+$ and for the time variable $\eta = \log D_+$, the linearized motion equations indeed read

$$\frac{\partial}{\partial\eta}\delta(\mathbf{x},\eta) - \hat{\theta}(\mathbf{x},\eta) = 0 \tag{47}$$

$$\frac{\partial}{\partial\eta}\hat{\theta}(\mathbf{x},\eta) + \left(\frac{3}{2}\frac{\Omega_m}{f_+^2} - 1\right)\hat{\theta} - \frac{3}{2}\frac{\Omega_m}{f_+^2}\delta(\mathbf{x},\eta) = 0, \tag{48}$$

Linear theory: The general background case

It can be rewritten as

$$\frac{\partial}{\partial \eta} \Psi_a(\mathbf{x}, \eta) + \Omega_a{}^b(\eta) \Psi_b(\mathbf{x}, \eta) = 0, \tag{49}$$

with

$$\Omega_a{}^b(\eta) = \begin{pmatrix} 0 & -1 \\ -\frac{3}{2}\frac{\Omega_m}{f_+^2} & \frac{3}{2}\frac{\Omega_m}{f_+^2} - 1 \end{pmatrix}. \tag{50}$$

A field representation of the nonlinear motion equations

Inserting the non-linear terms one eventually gets,

$$\frac{\partial}{\partial\eta}\Psi_a(\mathbf{k},\eta) + \Omega_a{}^b(\eta)\Psi_b(\mathbf{k},\eta) \quad = \quad \gamma_a{}^{bc}(\mathbf{k}_1,\mathbf{k}_2)\ \Psi_b(\mathbf{k}_1,\eta)\ \Psi_c(\mathbf{k}_2,\eta), \tag{73}$$

where $\Omega_a{}^b(\eta)$ is defined in eqn (50) and where (and that will be the case henceforth) we use the convention that repeated Fourier arguments are integrated over and the Einstein convention on repeated indices, and where the *symmetrized vertex* matrix γ_a^{bc} describes the non linear interactions between different Fourier modes.

A field representation of the nonlinear motion equations

The components of γ_a^{bc} are given by

$$\begin{aligned}
\gamma_2{}^{22}(\mathbf{k}_1, \mathbf{k}_2) &= \delta_{\rm Dirac}(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)\ \frac{|\mathbf{k}_1 + \mathbf{k}_2|^2 (\mathbf{k}_1 \cdot \mathbf{k}_2)}{2 k_1^2 k_2^2}, \\
\gamma_1{}^{21}(\mathbf{k}_1, \mathbf{k}_2) &= \delta_{\rm Dirac}(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)\ \frac{(\mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{k}_1}{2 k_1^2},
\end{aligned} \tag{74}$$

$\gamma_a{}^{bc}(\mathbf{k}_1, \mathbf{k}_2) = \gamma_a{}^{cb}(\mathbf{k}_2, \mathbf{k}_1)$, and $\gamma = 0$ otherwise, where $\delta_{\rm Dirac}$ denotes the Dirac distribution function. The matrix $\gamma_a{}^{bc}$ is independent on time (and on the background evolution) and encodes all the non-linear couplings of the system.

A field representation of the nonlinear motion equations

One can then take advantage of the knowledge of the Green function of this system to write a formal solution of eqn (73) as

$$\begin{aligned}\Psi_a(\mathbf{k},\eta) &= g_a{}^b(\eta)\,\Psi_b(\mathbf{k},\eta_0) + \\ &+ \int_{\eta_0}^{\eta} \mathrm{d}\eta'\; g_a{}^b(\eta,\eta')\;\gamma_b{}^{cd}(\mathbf{k}_1,\mathbf{k}_2)\Psi_c(\mathbf{k}_1,\eta')\Psi_d(\mathbf{k}_2,\eta'),\end{aligned} \tag{75}$$

where $\Psi_a(\mathbf{k},\eta_0)$ denotes the initial conditions.
In the following calculations we will be using the value of the $\Omega_a{}^b$ matrix to be that of the Einstein de Sitter background that is effectively assuming that D_- scales like $D_+^{-3/2}$. This is known to be a very good approximation even in the context of a $\Lambda-$CDM universe.

Diagrammatic representations

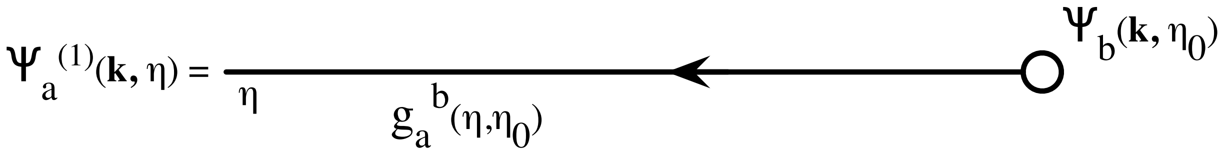


Figure: Diagrammatic representation of the linear propagator. Ψ_b represents the initial conditions and ${g_a}^b$ is the time dependent propagator. This diagram value is the linear solution of the motion equation.

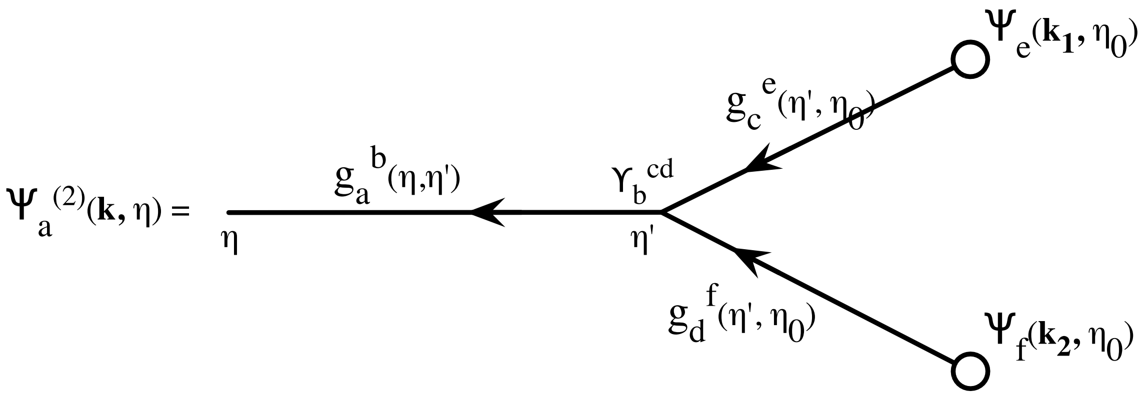


Figure: Diagrammatic representation of the fields at second order. This diagram value is given by eqn (75) when one replaces Ψ_c and Ψ_d in the second term of the right hand side by their linear expressions. In the diagram, each time one encounters a vertex, a time integration and a Dirac function in the wave modes is implicitly assumed.

Diagrammatic representations

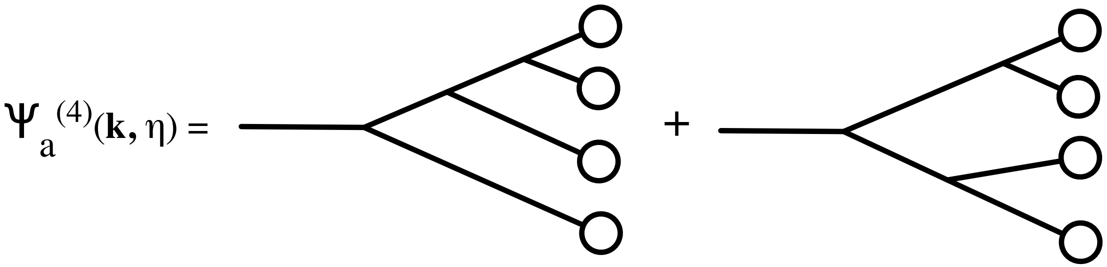


Figure: Diagrammatic representation of the fields at fourth order. Three different diagrams are found to contribute.

One of a nice feature of eqn (75) is that it admits simple diagrammatic representations in a way very similar to Feynman diagrams.

A field representation of the nonlinear motion equations

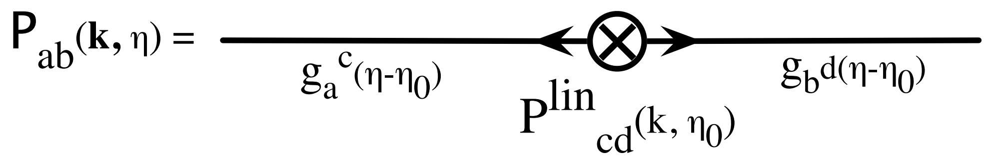


Figure: Diagrammatic representation of the power spectrum at linear order. The symbol $\otimes$ represents the linear power spectrum in the (adiabatic) growing mode.

A field representation of the nonlinear motion equations

Relevant statistical quantities are obtained however once ensemble average are taken. Assuming Gaussian initial conditions, one then can apply the Wick theorem to all the factors representing the initial field values that appear in diagrams (or product of diagrams) of interest. In practice, at least for these notes, the diagrams will all be computed assuming the initial conditions correspond effectively to the adiabatic linear growing mode. The simplest of such diagram is presented on Fig. 11. It corresponds to the ensemble average of $\langle \Psi_a(\mathbf{k}, \eta)\Psi_b(\mathbf{k}, \eta)\rangle$ and it makes intervene the linear power spectrum represented by $\otimes$. The previous construction can obviously be extended to any number of fields. The next diagrams will inevitably make intervene loops (in their diagram representation). One idea we will pursue here is to take advantage of such expansions to explore the density spectrum at 1-loop order and 2-loop order, also called at Next-to Leading Order and Next to Next to Leading Order (respectively NLO and NNLO).

A field representation of the nonlinear motion equations

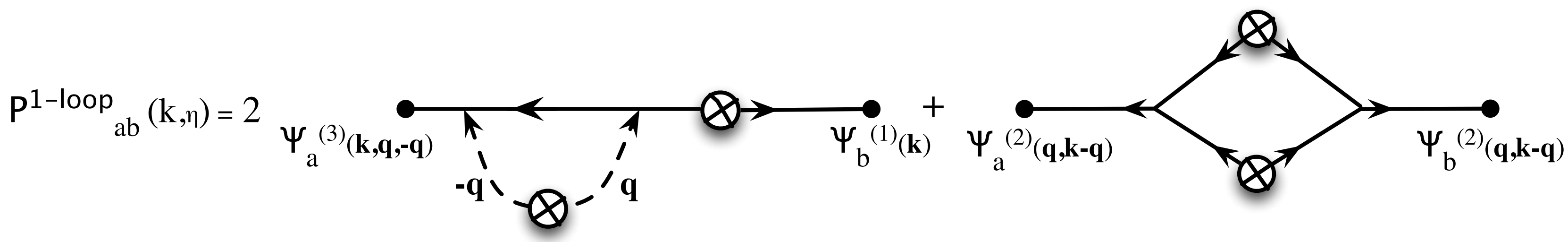


Figure: Diagrammatic representation of the 2 terms contributing to the power spectrum at one-loop order.

The 2 diagrams contributing to the power spectrum at NLO are shown on Fig. 12. The first calculation of such contributions were done in the 90's.

Scaling of solutions

It is interesting to compute the way subsequent orders in perturbation theory scale with the linear solution. As the vertices and the time integrations are both dimensionless operations, one can easily show that the $p-$th order expression of the density field is of the order of the power p of the linear density field. In other words, there are kernels functions $F_a^{(n)}$ such that

$$\Psi_a^{(p)}(\mathbf{k},\eta) = \int \frac{\mathrm{d}\mathbf{k}_1}{(2\pi)^{3/2}} \dots \frac{\mathrm{d}\mathbf{k}_p}{(2\pi)^{3/2}} \delta_{\mathrm{Dirac}}(\mathbf{k}-\mathbf{k}_{1\dots p}) \times \mathcal{F}_a^{(p)}(\mathbf{k}_1,\dots,\mathbf{k}_p;\eta)\delta_+(\mathbf{k}_1,\eta)\dots\delta_+(\mathbf{k}_p,\eta) \quad (76)$$

where $\delta_+(\mathbf{k}_i,\eta)$ is the linear growing mode for wave modes $\mathbf{k}_i$, $\mathbf{k}_{1\dots p} = \mathbf{k}_1 + \dots + \mathbf{k}_p$ and where $\mathcal{F}_a^{(p)}(\mathbf{k}_1,\dots,\mathbf{k}_p;\eta)$ are dimensionless functions of the wave modes and are a priori time dependent.

Scaling of solutions

For an Einstein-de Sitter background, the functions $\mathcal{F}_a^{(p)}(\mathbf{k}_1, \ldots, \mathbf{k}_p; \eta)$ are actually time independent and in general depends only very weakly on time (and henceforth on the cosmological parameters.) The functions $\mathcal{F}_a^{(p)}$ are usually noted F_p and G_p for respectively $a = 1$ and $a = 2$. For instance it is easy to show that

$$F_2(\mathbf{k}_1, \mathbf{k}_2) = \frac{5}{7} + \frac{1}{2}\frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{k_1^2} + \frac{1}{2}\frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{k_2^2} + \frac{2}{7}\frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)^2}{k_1^2\, k_2^2} \tag{77}$$

for an Einstein-de Sitter universe. For an arbitrary background the coefficient 5/7 and 2/7 are slightly altered but only very weakly.

Scaling of solutions

It can be noted that this kernel is very general and is actually directly observable. Indeed for Gaussian initial conditions the first non-vanishing contribution to the bi-spectrum is obtained when one, and only one, factor is written at second order in the initial field,

$$\langle \delta(\mathbf{k}_1)\delta(\mathbf{k}_2)\delta(\mathbf{k}_3)\rangle_c = \langle \delta^{(1)}(\mathbf{k}_1)\delta^{(1)}(\mathbf{k}_2)\delta^{(2)}(\mathbf{k}_3)\rangle_c + \mathrm{sym.} \quad (78)$$

and it is easy to show that it eventually reads

$$\begin{aligned} \langle \delta(\mathbf{k}_1)\delta(\mathbf{k}_2)\delta(\mathbf{k}_3)\rangle_c &= \delta_{\mathrm{Dirac}}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3) \\ &\times \left[2\,F_2(\mathbf{k}_1,\mathbf{k}_2)P^{\mathrm{lin.}}(k_1)P^{\mathrm{lin.}}(k_2) + \mathrm{sym.}\right] \end{aligned} \quad (79)$$

where sym. refers to 2 extra terms obtained by circular changes of the indices. The important consequence of this form is that the bispectrum therefore scales like the square of the power spectrum.

Scaling of solutions

In particular the reduced bispectrum defined as

$$Q(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \frac{B(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)}{P(k_1)P(k_2) + P(k_2)P(k_3) + P(k_3)P(k_1)} \quad (80)$$

is expected to have a time independent amplitude at early time.

Scaling of solutions

More generally, the connected p-point correlators at lowest order in perturbation theory scale like the power $p-1$ of the 2-point correlators: it comes from the fact that in order to connect p points using the Wick theorem one needs at least $p-1$ lines connecting a product of $2(p-1)$ fields taken at linear order. For instance the local $p-$order cumulant of the local density contrast $\langle\delta^p\rangle_c$ scales like,

$$\langle\delta^p\rangle_c \sim \langle\delta^2\rangle^{p-1}. \quad (81)$$

In the last sections of these notes, more detailed presentation of these relations will be given.

Scaling of solutions

Other consequences of these scaling results concern the p-loop corrections to the power spectrum. Indeed one expects to have

$$P^{\mathrm{p-loop}}(k) \sim P^{\mathrm{lin.}}(k) \left[\int \frac{\mathrm{d}q}{q}\, q^3 P(q)\right]^p. \tag{82}$$

At least that would be the case if the linear power spectrum peaked at wave modes about k. This is not necessarily the case. In the following we explore the mode coupling structure, i.e. how modes q are contributing to the corrective terms to the power spectrum depending on whether they are much smaller (infrared domain) or much larger (ultra-violet domain) than k.